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STRESS FUNCTIONS IN MULTIPLY CONNECTED VOLUMES†

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It is shown that stress functions exist in a spatially multiply connected volume only if the stresses are such that the total effect of each cavity on the volume vanishes. The general form of an arbitrary statically admissible stress field in a multiply connected body is found. The non-uniqueness of the stress function tensor for a given stress field is investigated. On this basis the general expression for statically admissible stress fields in a multiply connected body that satisfy the zero boundary conditions on a part of the surface of the body is obtained.

1. IT IS well known [1–3] that a stress field $\boldsymbol{\sigma}$ that satisfies the equilibrium conditions

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0} \tag{1.1}$$

in the volume V can be expressed in terms of stress functions. This expression was subsequently generalized [4] to the case of a stress tensor with integrable components, from which, in particular, it follows that the stress functions exist even if $\boldsymbol{\sigma}$ is discontinuous and satisfies the additional equilibrium conditions

$$\mathbf{v} \cdot \boldsymbol{\sigma}^+ = \mathbf{v} \cdot \boldsymbol{\sigma}^- \text{ on } S_p \tag{1.2}$$

on the discontinuity surface S_p. Here $\boldsymbol{\nu}$ is the normal vector to S_p, and $\boldsymbol{\sigma}^+$, $\boldsymbol{\sigma}^-$ are the limiting values of $\boldsymbol{\sigma}$, which can be obtained by letting $\mathbf{x}$ approach S_p from the opposite sides. (All the surfaces considered in this paper as well as the lines constituting their boundaries are assumed to be smooth and regular [5], and all the functions are assumed to have continuous derivatives of any order used at every point, except perhaps on some surfaces of discontinuity, in which case it is assumed that the functions and their derivatives have finite limits as the surface of discontinuity is approached from either side.) The following invariant assertion [6–8] has been formulated by introducing the stress function tensor $\boldsymbol{\varphi}$: Eqs (1.1) and (1.2) are satisfied if and only if

$$\boldsymbol{\sigma} = \nabla \times (\nabla \times \boldsymbol{\varphi})^* = \mathbf{Ink}\, \boldsymbol{\varphi} \tag{1.3}$$

We remark that in all the articles mentioned above V has been assumed to be a simply connected volume. Apparently, the problem of the existence of stress functions in multiply connected volumes has not been considered so far.

We shall first show that, generally speaking, Eq. (1.3) does not follow from (1.1) and (1.2) in the case of a multiply connected body, i.e. for the stress function to exist, one must, in fact, impose additional restrictions on $\boldsymbol{\sigma}$. We will assume that (1.3) is satisfied. Then, by Stokes' formula, we find that

$$\mathbf{F}(\boldsymbol{\sigma}, S) = \iint_S \mathbf{ds} \cdot \boldsymbol{\sigma} = \iint_S \mathbf{ds} \cdot (\nabla \times (\nabla \times \boldsymbol{\varphi})^*) = \oint_{\partial S} \mathbf{dx} \cdot (\nabla \times \boldsymbol{\varphi})^* \tag{1.4}$$

where $S \subset V$ and where $\mathbf{F}(\boldsymbol{\sigma}, S)$ is the total force acting on the side of S determined by the terminal point of the vector $\mathbf{ds}$. The orientation of the contour ∂S used to evaluate the curvilinear integral

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corresponds in a standard manner to the direction of $\mathbf{ds}$. In particular, if S is a closed surface, it follows from (1.4) that $\mathbf{F}(\boldsymbol{\sigma}, S) = \mathbf{0}$ for any stress field $\boldsymbol{\sigma}$. Now let $\boldsymbol{\sigma}$ be a stress field in a spatially multiply connected body V with a cavity and a force $\mathbf{F}_1 \neq \mathbf{0}$ acting on V from within the cavity, and let S be a closed surface containing the cavity. If (1.3) were satisfied, then, from what has been said above and the fact that the part of V cut off by S is in a state of equilibrium, it would follow that $\mathbf{F}_1 = \mathbf{0}$, contrary to our assumption.

The conditions that must be satisfied by the stress field $\boldsymbol{\sigma}$ in order that the stress function tensor exists yield the following theorem.

Theorem 1. Let the cavities contained in a spatially multiply connected body V with multiply connected surface be bound by closed surfaces S_i, $i = 1, \ldots, n$ such that $S_i \cap S_j = \emptyset$ for $i \neq j$.

Then a stress field that satisfied (1.1) and (1.2) can be represented in the form (1.3) if and only if

$$\mathbf{F}(\boldsymbol{\sigma}, S_i) = \mathbf{0}, \ \mathbf{M}(\boldsymbol{\sigma}, S_i) = \mathbf{0}, \ i = 1, \ldots, n \tag{1.5}$$

The function $\mathbf{F}(\boldsymbol{\sigma}, S)$ is defined by (1.4) and $\mathbf{M}(\boldsymbol{\sigma}, S)$ is the total movement (with respect to a fixed point) of the forces acting on one side of the surface $S \subset V$.

Proof. The necessity of the first equality in (1.5) follows from the above discussion. The necessity of the second equality can be proved in the same way.

We shall prove that conditions (1.5) are sufficient. To do so, we shall show that $\boldsymbol{\sigma}$ can be extended to a simply connected volume $V^c \supset V$ in such a way that conditions (1.1) and (1.2) are preserved, and to the field on V^c we shall apply the results on the existence of stress functions in simply connected volumes [4].

We shall consider a cavity V_i bounded by a surface S_i as a rigid body with a load $\boldsymbol{\nu} \cdot \boldsymbol{\sigma}$ distributed on the surface. By (1.5), the load is self-balanced. Thus the stress field $\boldsymbol{\sigma}_i$ established inside V_i will satisfy (1.1). The body $V' = V \cup V_1 \cup \ldots \cup V_n$ is spatially simply connected and the stress field $\boldsymbol{\sigma}'$ defined in it so that it coincides with $\boldsymbol{\sigma}$ on V and with $\boldsymbol{\sigma}_i$ in V_i, satisfies (1.1) and (1.2).

If the surface of V' is also simply connected, then the proof is completed. Otherwise, let V^c be a simply connected volume containing V', $V^c \supset V'$. Since V' is spatially simply connected, it follows that $V'' = V^c \backslash V'$ is connected and its surface $\partial V''$ consists of the surface $\partial V'$ of V' and the remaining part $S' = \partial V'' \backslash \partial V'$ such that $S' \neq \emptyset$. We regard V'' as an elastic body with the load $\boldsymbol{\nu} \cdot \boldsymbol{\sigma}$ (generally speaking, not self-balanced) distributed over the surface $\partial V'$ and with an additional load on S' such that the total load on $\partial V''$ (for example, attached to S') is balanced. Let $\boldsymbol{\sigma}''$ be the stress field in V''. The stress field $\boldsymbol{\sigma}^c$ defined inside the simply connected body V^c and identical with $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}''$ on V' and V'', respectively, satisfies conditions (1.1) and (1.2) and constitutes the desired extension of $\boldsymbol{\sigma}$.

As can be seen from Theorem 1, the fact that the surface of a body is multiply connected does not prevent the existence of stress functions. Restrictions (1.5) appear only when V contains cavities. In this case the general form of the stress fields satisfying (1.1) and (1.2) can be obtained by adding to (1.3) some terms that compensate for the influence of each cavity V_i on V. Namely, with the stress field $\boldsymbol{\sigma}$ we associate the $6n$-dimensional vector

$$\mathbf{N}(\boldsymbol{\sigma}) = (\mathbf{F}(\boldsymbol{\sigma}, S_1); \mathbf{M}(\boldsymbol{\sigma}, S_1); \ldots \mathbf{F}(\boldsymbol{\sigma}, S_n); \mathbf{M}(\boldsymbol{\sigma}, S_n))$$

and we assume that there are known tensor fields $\boldsymbol{\sigma}_j$ ($j = 1, \ldots, 6n$) satisfying (1.1) and (1.2) such that the corresponding vectors $\mathbf{N}(\boldsymbol{\sigma}_j)$ are linearly independent. From Theorem 1 we obtain the following corollary.

Corollary 1. A symmetric tensor field $\boldsymbol{\sigma}$ in a spatially $(n+1)$-connected volume with multiply connected surfaace satisfies (1.1) and (1.2) if and only if

$$\sigma = \operatorname{Ink} \varphi + \sum_{j=1}^{6n} c_j \sigma_j \tag{1.6}$$

for some symmetric tensor φ and some numbers c_j.

2. By means of (1.3), φ can be determined from a given tensor σ in a non-unique way, since the equality

$$\operatorname{Ink} \varphi \equiv 0 \text{ on } V \tag{2.1}$$

is satisfied for any non-zero tensor φ of the form

$$\varphi = {}^1/_2 [\nabla \mathbf{q} + (\nabla \mathbf{q})^*] = \operatorname{def} \mathbf{q} \tag{2.2}$$

where $\mathbf{q}$ is an arbitrary vector field on V.

If V is a simply connected volume, then the kernel of the operator Ink can be completely described by means of tensors of the form (2.2) [6].

Let us investigate the structure of this kernel in the multiply connected case.

Let L be an oriented curve. We set

$$\mathbf{p}(L, \varphi) = \int d\mathbf{x} \cdot (\nabla \times \varphi)^*$$
$$\mathbf{q}(L, \varphi) = \int d\mathbf{x} \cdot (\varphi - (\mathbf{x}_1 - \mathbf{x}) \times (\nabla \times \varphi))^*, \quad \lambda(L, \varphi) = (\mathbf{p}, \mathbf{q}) \tag{2.3}$$

where $\mathbf{x}_1$ is a fixed point and $\mathbf{x}$ is a "moving" point (integration is carried out with respect to the latter point). The integrals in (2.3) are evaluated over the curve L in the direction determined by its orientation.

Lemma 1. The tensor φ can be represented in the form (2.2) if and only if

$$\lambda(L, \varphi) = 0 \tag{2.4}$$

for any closed curve $L \subset V$.

Proof. We shall make use of a geometric analogy [6]. If we regard $\mathbf{q}$ as the displacement vector and φ as the tensor of small deformations, then (2.1) will serve as the compatibility condition for φ and (2.3) will be the Cesàro formulas, which reconstruct the displacement vector $\mathbf{q}$ and its rotation $-2\mathbf{p}$ from prescribed deformations. Hence the assertion of the lemma follows immediately.

From the above geometric analogy it also follows that, for a body with multiply connected surface (for example a torus), the kernel of the operator Ink is larger than the set of tensors of the form (2.2), since in this case the local compatibility of the deformations does not, in general, imply global compatibility.

In the subsequent analysis we shall need the following definitions.

We shall say that closed oriented curves L' and L'' are equivalent in a volume V if, by means of a continuous deformation, L' can be combined with L'' (so that their orientations coincide) or with a curve L_e'' that differs from L'' only by segments that are traversed the same number of times in opposite directions. The equivalence classes generated by this relation will be referred to as cycles on V and denoted by G.

If two cycles G' and G'' coincide with each other, we shall write $G' \overline{\overline{{}_V}} G''$. The cycle G consisting of curves equivalent to ones that are contractible to a point in V will be called the zero cycle. We shall write $G \overline{\overline{{}_V}} 0$. The cycle consisting of the same closed curves as G but with opposite orientation will be denoted by $-G$.

We introduce the notion of the sum of two cycles G_1 and G_2. To do so, we note that, since V is connected, G_1 and G_2 always contain curves L_1 and L_2 with a common segment L' traversed in opposite directions. We consider an oriented curve L'' that can be obtained from L_1 and L_2 by discarding the common part L' and L_1 and L_2 so that the orientation of the curves is preserved. The cycle generated by L'' will be called the sum $G_1 + G_2$.

The cycle $G' + (-G'')$ will be called the difference $G' - G''$. The product kG of a cycle and an integral number will be defined as the sum (the difference) of the given number of cycles.

One can verify that the above operations turn the set of cycles into a linear space over the ring Z of integral numbers (which is isomorphic to one-dimensional homologies of V).

A system of cycles $\{G_k\}_{k=1}^{m}$ is called a basis in the space of cycles on V if any cycle is a linear combination of G_k and none of the cycles G_k is a linear combination of the other cycles belonging to the basis.

A spatially multiply connected body V (of finite dimensions) with multiply connected surface is, in general, bounded by an $(n+1)$-connected surface S_i ($S_i \cap S_j = \emptyset$ for $i \neq j$), where n is the number of cavities in the body, each of the surfaces S_i being a connected two-dimensional oriented manifold without a boundary. Starting from the fact that each of the surfaces S_i is homeomorphic to a sphere with r_i handles [9], one can prove that there exists a basis in the space of cycles (and $m = r_1 + \ldots + r_n$).

We shall now prove that if condition (2.1) is satisfied, then the vector-valued function $\boldsymbol{\lambda}$ defined by (2.3) is constant on closed curves belonging to the same cycle and so it can be regarded as a function $\boldsymbol{\lambda}(G, \boldsymbol{\varphi})$ defined on the set of cycles.

Indeed, let L', $L'' \in G$ and let $S \subset V$ be a surface on which the curve L' can be deformed into L_e''. Applying formula (1.4) to the doubly-connected surface S and using (2.1) and (2.3), we get

$$0 = \oint_{\partial S} d\mathbf{x} \cdot (\nabla \times \boldsymbol{\varphi})^* = \mathbf{p}(L', \boldsymbol{\varphi}) - \mathbf{p}(L_e'', \boldsymbol{\varphi}) \tag{2.5}$$

In (2.5) it is taken into account that $\partial S = L' \subset L''$ and the direction in which ∂S is traversed when evaluating the curvilinear integral over one of the curves L' or L_e'' (for example, L_e'') is opposite to its orientation. By analogy, one can also derive the other required equality $\mathbf{q}(L', \boldsymbol{\varphi}) = \mathbf{q}(L'', \boldsymbol{\varphi})$.

The vector-valued function $\boldsymbol{\lambda}(G, \boldsymbol{\varphi})$ obtained in this way, which is defined on the set of cycles and tensor fields that satisfy (2.1), turns out to be linear both in G and $\boldsymbol{\varphi}$.

Lemma 2. let $\{G_k\}_{k=1}^{m}$ be a basis in the space of cycles on V, and let $\mathbf{a}_k = (\mathbf{p}_k, \mathbf{q}_k)$, where $k = 1, \ldots, m$, be an arbitrary system of fixed vectors in R^6 ($\mathbf{p}_k, \mathbf{q}_k \in R^3$).

Then there exists a symmetric tensor field $\boldsymbol{\varphi}$ that satisfies (2.1) such that

$$\boldsymbol{\lambda}(G_k, \boldsymbol{\varphi}) = \mathbf{a}_k, \quad k = 1, \ldots, m \tag{2.6}$$

Proof. We fix $i (1 \leq i \leq m)$ and consider a torus $V_i \supseteq V$ such that $G_i \underset{V_i}{\neq} 0$, and $G_j \underset{V_i}{=} 0$ for $j \neq i$. It is possible to construct such a torus, since the surface of v is homeomorphic to a sphere with handles and V_i can be obtained by "attaching" additional handles.

Let $L \in G_i$ and let the surface S cut V_i into a simply connected body and intersect L at a single point. We denote by V_ε^+ and V_ε^- the ε-layers of V_i adjacent to the opposite sides of S, and we define a vector-valued function $\mathbf{q}_i(\mathbf{x})$ ($\mathbf{q}_i \in R^3$) as follows:

$$\mathbf{q}_i(\mathbf{x}) = \begin{cases} 0 & \text{for } \mathbf{x} \in V_\varepsilon^- \\ \mathbf{q}_i - \frac{1}{2}\mathbf{p}_i \times \mathbf{x} & \text{for } \mathbf{x} \in V_\varepsilon^+ \end{cases}$$

For the remaining $\mathbf{x} \in V_i$ the function $\mathbf{q}_i(\mathbf{x})$ can be defined as an arbitrary smooth function. (If $\mathbf{q}_i(\mathbf{x})$ is regarded as the displacement vector, then the layer V_ε^- is immovable and the whole layer V_ε^+ is displaced and rotated as a rigid body.) We define $\boldsymbol{\varphi}_i$ to be the strain tensor associated with the displacement $\mathbf{q}_i$ by means of the Cauchy formulas (2.2). Then Ink $\boldsymbol{\varphi}_i \equiv 0$, and, by virtue of the above geometric analogy,

$$\boldsymbol{\lambda}(L, \boldsymbol{\varphi}_i) = \mathbf{a}_i \text{ and } \boldsymbol{\lambda}(L', \boldsymbol{\varphi}_i) = 0, \text{ if } L' \in G_j \text{ for } j \neq i.$$

Carrying out the same construction for each i and setting $\boldsymbol{\varphi} = \boldsymbol{\varphi}_1 + \ldots + \boldsymbol{\varphi}_m$, we can obtain the desired result.

The assertion below, which describes the kernel of Ink in the multiply connected case, follows from Lemmas 1 and 2.

Corollary 2. Let $\{G_k\}_{k=1}^m$ be a basis in the space of cycles on V, let $\boldsymbol{\Lambda}(\boldsymbol{\varphi}) = \lambda(G_1, \boldsymbol{\varphi}), \ldots, \lambda(G_m, \boldsymbol{\varphi})$ and let $\boldsymbol{\varphi}_i$ $(i = 1, \ldots, 6m)$ be tensor fields that satisfy (2.1) such that the corresponding $6m$-dimensional vectors $\boldsymbol{\Lambda}(\boldsymbol{\varphi}_i)$ are linearly independent.

Then $\boldsymbol{\varphi}$ satisfies (2.1) if and only if

$$\boldsymbol{\varphi} = \mathbf{def}\,\mathbf{q} + c_1\boldsymbol{\varphi}_1 + \ldots + c_{6m}\boldsymbol{\varphi}_{6m}$$

for some vector field $\mathbf{q}$ and some numbers c_i.

3. We will consider the problem of expressing the general form of a stress field that satisfies (1.1), (1.2), and the additional condition

$$\boldsymbol{\nu}\cdot\boldsymbol{\sigma} = 0 \text{ on } S_F \subseteq \partial V \tag{3.1}$$

where ∂V is the surface of the multiply connected body V, in terms of the stress function tensor (1.3).

For a simply connected volume and the case $S_F = \partial V$, it has been shown [10] that the general solution of the system of equations (1.1), (1.2) and (3.1) can be obtained in the form

$$\boldsymbol{\sigma} = \mathbf{Ink}\,(\omega^2\boldsymbol{\psi}) \tag{3.2}$$

where $\boldsymbol{\psi}$ is an arbitrary symmetric tensor and the (fixed) function $\omega(\mathbf{x})$ satisfies the conditions

$$\begin{aligned} \omega(\mathbf{x}) = 0,\ \partial\omega(\mathbf{x})/\partial\boldsymbol{\nu} \neq 0 \quad &\text{for} \quad \mathbf{x} \in S_F \\ \omega(\mathbf{x}) > 0 \quad &\text{for} \quad \mathbf{x} \in V \setminus S_F \end{aligned} \tag{3.3}$$

If $S_F \neq \partial V$, then, generally speaking, (3.2) is not the general solution even if V is simply connected. Indeed, for example, let V be a cylinder that is being stretched along its axis and whose lateral surface S_F is free of any loads, and let S be a cross-section of the cylinder with boundary ∂S lying on S_F, $\partial S \subset S_F$. If the representation (3.2) were satisfied [i.e. $\boldsymbol{\varphi} = \omega^2\boldsymbol{\psi}$ in (1.3)], then it would follow from the first equality in (3.3) that the integral on the right-hand side of (1.4) vanishes and the total force on S is equal to zero, contrary to the assumption that the cylinder is being stretched. In a similar manner one can obtain the stress field inside a torus, the entire surface of which is free of any loads. The latter stress field cannot be expressed in the form (3.2).

The following theorem describes the set of those stress fields in a multiply connected volume V that can be expressed in the form (3.2).

Theorem 2. Let $S_U = \partial V \backslash S_F$ be a part of the smooth surface ∂V of a body V of finite dimensions consisting of pairwise disjoint connected components S_i, i.e. let $S_U = S_1 \cup S_2 \cup \ldots \cup S_r$, where $S_i \cap S_j = \varnothing$ for $i \neq j$. Let $S_k' \subset V$ $(k = 1, \ldots, t)$ be connected surfaces bounded by $\partial S_k' \subset S_F$ such that the intersection of any of them with V is a connected set,† but the addition of an intersecting surface $S' \subset V$ such that $\partial S' \subset S_F$ turns the set into a disconnected one, and let

$$\begin{aligned} \mathbf{N}(\boldsymbol{\sigma}) = (&\mathbf{F}(\boldsymbol{\sigma}, S_1), \mathbf{M}(\boldsymbol{\sigma}, S_1), \ldots, \mathbf{F}(\boldsymbol{\sigma}, S_r), \mathbf{M}(\boldsymbol{\sigma}, S_r), \mathbf{F}(\boldsymbol{\sigma}, S_1'), \\ &\mathbf{M}(\boldsymbol{\sigma}, S_1'), \ldots, \mathbf{F}(\boldsymbol{\sigma}, S_t'), \mathbf{M}(\boldsymbol{\sigma}, S_t')) \end{aligned} \tag{3.4}$$

where $\mathbf{F}(\boldsymbol{\sigma}, S)$ and $\mathbf{M}(\boldsymbol{\sigma}, S)$ are defined by (1.4) and (1.5).

Then a stress field $\boldsymbol{\sigma}$ that satisfies (1.1), (1.2) and (3.1) can be represented in the form (3.2) if and only if

$$\mathbf{N}(\boldsymbol{\sigma}) = 0 \tag{3.5}$$

Proof. The sufficiency can be proved as in the simply connected case in [1].

We shall prove the necessity. Let $\boldsymbol{\sigma}$ satisfy (1.1), (1.2), (3.1) and (3.5). Since condition (1.5) of Theorem 1 follows from (3.1) and (3.5), there exists a tensor $\boldsymbol{\varphi}'$ such that (1.3) is satisfied.

†If $S_U = \varnothing$ and/or no surfaces S_k satisfying the given conditions exist, then we set $r = 0$ and/or $t = 0$, respectively. For $r = t = 0$, condition (3.5) vanishes.

Let L', $L'' \subset S_F$ be closed oriented curves. It can be shown that if L' and L'' belong to the same cycle on V, then

$$\boldsymbol{\lambda}\ (L', \boldsymbol{\varphi}') = \boldsymbol{\lambda}\ (L'', \boldsymbol{\varphi}') \tag{3.6}$$

Let $S' \subset V$, $S' \not\subset \partial V$ be a surface on which L' can be deformed into L_e''. By the assumption of the theorem, the surfaces S' and S_k' cut V into disconnected parts V' and V''. For example, we shall consider V'. The surface of V' consists of S', a part of S_F, and some of the surfaces S_i and S_k'. Since the total forces and torques acting on the surfaces S_i and S_k' and on any part of S_F are equal to zero, it follows from the equilibrium condition for V' that the force and torque acting on S' are both equal to zero. Next, repeating the argument that follows formula (2.5), we can obtain (3.6).

Thus, if only the curves lying on S_F are regarded as the elements of a cycle, then the vector-valued function $\boldsymbol{\lambda}$ defined by (2.3) can also be regarded as a function of the cycles (although in this case Ink $\boldsymbol{\varphi} \not\equiv \mathbf{0}$).

Let $\{G_k\}_{k=1}^{m}$ be a basis in the space of cycles and let $\mathbf{a}_k = \boldsymbol{\lambda}(G_k, \boldsymbol{\varphi}')$. By Lemma 2, there exists a tensor field $\boldsymbol{\varphi}''$ such that Ink $\boldsymbol{\varphi}'' \equiv \mathbf{0}$ and $\boldsymbol{\lambda}(G_k, \boldsymbol{\varphi}'') = -\mathbf{a}_k$. Then the total field $\boldsymbol{\varphi} = \boldsymbol{\varphi}' + \boldsymbol{\varphi}''$ is a stress function tensor determining the same stress field $\boldsymbol{\sigma}$ as $\boldsymbol{\varphi}'$ such that $\boldsymbol{\lambda}(G, \boldsymbol{\varphi}) = \mathbf{0}$ for any closed curve L lying on S_F. The remaining part of the proof is the same as the proof in the simply connected case in [10] [starting from formula (2.6)].

We remark that in the assumptions of Theorem 2 it suffices to require that (3.5) be satisfied for any $(r-1)$ subsurfaces S_i, e.g. for $S_1, \ldots, S_{r-1}$, since from the equality $\mathbf{N}(\boldsymbol{\sigma}, S_1, \ldots, S_{r-1}) = \mathbf{0}$ and the fact that V is balanced as a whole it follows immediately that $\mathbf{N}(\boldsymbol{\sigma}, S_r) = \mathbf{0}$.

Let $r_1 = \max(r, 1)$. We assume that there are $6(r_1 + t - 1)$ known stress fields $\boldsymbol{\sigma}_i$ such that the vectors $\mathbf{N}(\boldsymbol{\sigma}_i)$ associated with them by means of (3.4) are linearly independent. Using Theorem 2, one can obtain the general solution of the problem under consideration in the following form.

Corollary 3. A stress field $\boldsymbol{\sigma}$ satisfies Eqs (1.1), (1.2) and (1.3) if and only if

$$\boldsymbol{\sigma} = \operatorname{Ink}(\omega^2 \boldsymbol{\psi}) + \sum_{i=1}^{6(r_1+t-1)} c_i \boldsymbol{\sigma}_i \tag{3.7}$$

for some symmetric tensor $\boldsymbol{\psi}$ and some numerical coefficients c_i.

Formula (3.7) can be used to construct admissible variations when solving the boundary value problems of deformable rigid body mechanics on the basis of Castigliano's variational principle. As opposed to the results of [10], this formula is applicable to multiply connected bodies even in the case when $S_n \neq \partial V$. The possibility of finding a variation of the form (3.7) by construction rests on the fact that the explicit formulas for the functions ω can be obtained by the method of R-functions [11] for regions of almost arbitrary form.

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